RESEARCH DEPARTMENT

Resolving the conflict between amplitude and group delay requirements for certain types of low-pass filter

TECHNOLOGICAL REPORT No. G-101

UDC 621·372·542·2

1966/9

THE BRITISH BROADCASTING CORPORATION
ENGINEERING DIVISION

RESEARCH DEPARTMENT

RESOLVING THE CONFLICT BETWEEN AMPLITUDE AND GROUP DELAY REQUIREMENTS FOR CERTAIN TYPES OF LOW-PASS FILTER

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SUMMARY

In applications to television, if a low-pass filter is required and the tolerances on variation of group delay and amplitude in the effective pass band are not stringent but simplicity is important, priority should be given to meeting the group-delay requirements. This is discussed in detail for cases in which the insertion transfer function is the reciprocal of a polynominal of degree 4; step responses, 2T-pulse responses and K-ratings for these cases are compared.

1. INTRODUCTION

Ideally a low-pass filter is required to have constant amplitude and group delay within the pass-band, and infinite attenuation outside this band. In theory at any rate, it is possible to design a minimum-phase lossless network for which specified restrictions on pass-band ripple, stop-band loss and limits of the pass- and stop-bands are prescribed. But there is no guarantee that such a network has satisfactory performance with respect to group-delay, and it may be necessary to complicate the network still further by adding all-pass phase-correcting networks which do not affect the modulus of the transfer function.

Much effort has been directed to design techniques in which close tolerances have to be met in the final result at the expense of lengthy calculation and considerable complexity of apparatus. Here, however, the objective is to see what compromise is possible between good performance with respect to amplitude and with respect to group delay when only minimum-phase networks of simple types are used.

We shall therefore assume that the transfer function of the filter is the reciprocal of a polynominal of degree 4 in p which strictly is the operator d/dt but for the purposes of this work can be taken as $j\omega$ at frequency $\omega/2\pi$ (j^2 being, as usual, equivalent to -1). If precedence is given to minimizing variations in the amplitude, we shall assume that the best possibility available is the well-known maximally-flat Butterworth case; if precedence is given to minimizing variations in the group delay, we shall assume that the best possibility available is one of the "quartically flat" set discussed by Gouriet.[1] The coefficients in the reciprocal of the transfer function are then varied linearly between their values for the Butterworth case and their values for the Gouriet case, and it soon becomes clear that the optimum available is very near the Gouriet case.

In Section 2, the amplitude and group delay are derived when the transfer function is the reciprocal of a general polynomial of degree 4, and the particular Butterworth and Gouriet cases are considered. In Section 3 the effect of a linear variation between the Gouriet and Butterworth coefficients is considered; this is illustrated in Figs. 1 and 2, and the relative merits of various particular cases are discussed. In Section 4, the step responses associated with the Butterworth case, the Gouriet case and a "compromise" case are discussed and shown in Figs. 3, 4 and 5. The corresponding K-ratings (in response to a 2T-pulse) were deduced by means of a computer programme. Section 5 gives conclusions.

2. DERIVATION OF AMPLITUDE AND GROUP DELAY FROM THE TRANSFER FUNCTION, WITH PARTICULAR REFERENCE TO THE GOURIET AND BUTTERWORTH CASES

It is assumed that the transfer function $f(p)$ is the reciprocal of a polynominal of degree 4, namely

$$f(p) = \frac{1}{1 + a_1p + a_2p^2 + a_3p^3 + a_4p^4} \qquad (1)$$

and the object is to choose a_1, a_2, a_3, a_4 so that the associated amplitude and group delay both vary as little as possible. Alternatively, the object might be expressed thus: to find out the best possible compromise between good amplitude/frequency

and group-delay/frequency constancy and good cut-off when the transfer function is the resiprocal of a quartic polynominal. The amplitude $A(\omega)$ corresponding to (1) is given by

$$A(\omega) = \{ f(j\omega)f(-j\omega) \}^{1/2} = \frac{1}{[(1 - a_2\omega^2 + a_4\omega^4)^2 + \omega^2(a_1 - a_3\omega^2)^2]^{1/2}} \qquad (2)$$

while the group delay τ is given by

$$\tau = \frac{d\phi}{d\omega} \text{ where } \tan\phi = \frac{\omega(a_1 - a_3\omega^2)}{1 - a_2\omega^2 + a_4\omega^4} \qquad (3)$$

so that

$$\tau = \frac{a_1 + (a_1a_2 - 3a_3)\omega^2 + (a_2a_3 - 3a_1a_4)\omega^4 + a_3a_4\omega^6}{(1 - a_2\omega^2 + a_4\omega^4)^2 + \omega^2(a_1 - a_3\omega^2)^2} \qquad (4)$$

If the filter having the insertion transfer function $f(p)$ given by Equation (1) above is to be realised by means of a passive network,* a_1, a_2, a_3 and a_4 must all be positive, and these quantities must also satisfy

$$a_1a_2a_3 > a_3^2 + a_4a_1^2 \qquad (5)$$

The two particular cases with which we are concerned are

(a) the Butterworth case, in which

$$1/f(p) = [1 + 2rp\sin(\pi/8) + r^2p^2]\,[1 + 2rp\sin(3\pi/8) + r^2p^2] \qquad (6)$$

* The networks whose transfer functions are given by Equation 6, 8 (with K = 0·44) and (11) (with λ = 0·009) below can all be realised as LC ladder networks with resistive terminations.

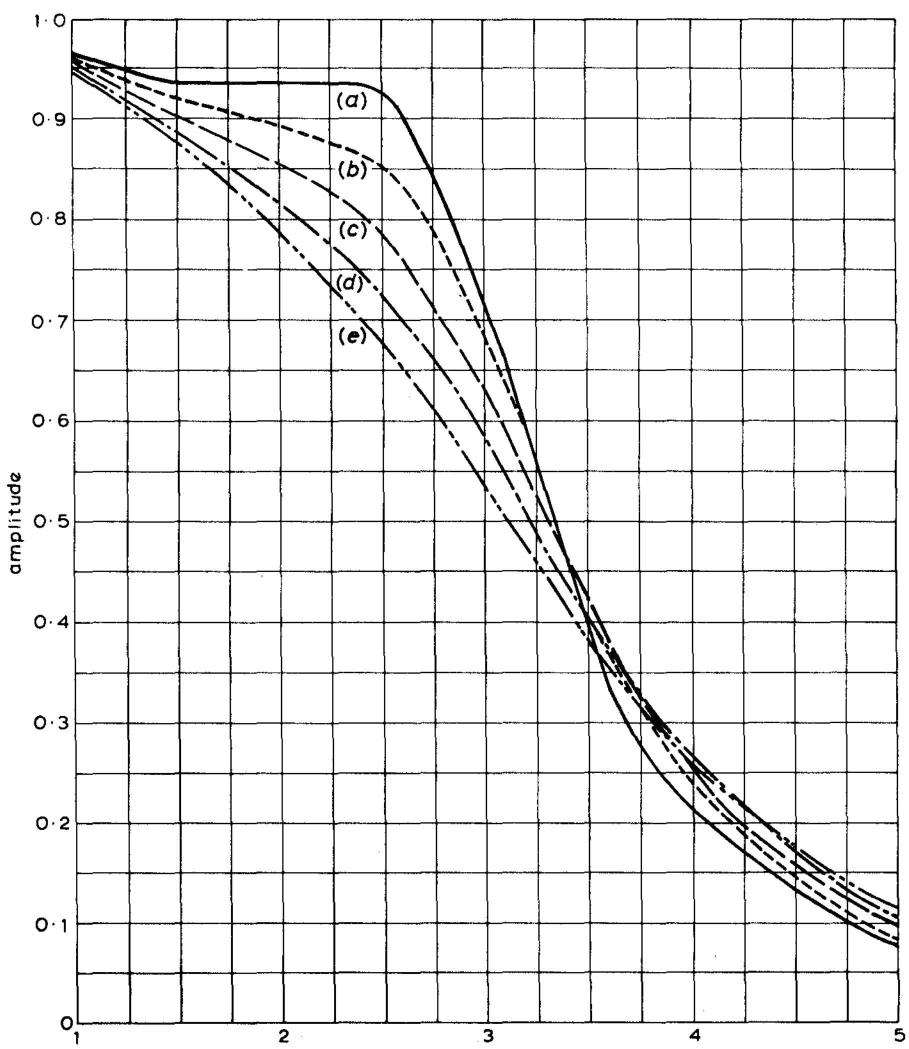

Fig. 1 - Variation of amplitude with frequency when transfer function is given by Equations (11) and (12)

(*a*) λ = 0·012 (*b*) λ = 0·009 (*c*) λ = 0·006 (*d*) λ = 0·003 (*e*) λ = 0

where r is a scale factor at our choice. The value of ω at the nominal cut-off frequency, when the amplitude is 3 dB down relative to its value at zero frequency is $1/r$, since

$$1/[f(\mathrm{j}\omega)f(-\mathrm{j}\omega)] = 1 + r^8\omega^8 \qquad (7)$$

(b) the Gouriet case, in which

$$1/f(\mathrm{p}) = 1 + \rho\mathrm{p} + K\rho^2\mathrm{p}^2 + (K - \tfrac{1}{3})\rho^3\mathrm{p}^3 + \tfrac{1}{3}(K - 0{\cdot}4)\rho^4\mathrm{p}^4 \qquad (8)$$

where again ρ is a mere scale factor at our choice, and K is a parameter controlling the shape of the group-delay/frequency characteristic, and the corresponding group delay τ is given by

$$\tau = \rho\,\frac{1 + c_2\rho^2\omega^2 + c_4\rho^4\omega^4 + c_6\rho^6\omega^6}{1 + c_2\rho^2\omega^2 + c_4\rho^4\omega^4 + b_6\rho^6\omega^6 + b_8\rho^8\omega^8} \qquad (9)$$

where

$$\left.\begin{aligned} c_2 &= 1 - 2K & b_6 &= \tfrac{1}{3}K^2 - 0{\cdot}4K + \tfrac{1}{9} \\ c_4 &= (15K^2 - 20K + 6)/15 & b_8 &= \tfrac{1}{9}(K - 0{\cdot}4)^2 \\ c_6 &= \tfrac{1}{3}(K - \tfrac{1}{3})\,(K - 0{\cdot}4) & & \end{aligned}\right\} \qquad (10)$$

and K is at our choice. For $f(\mathrm{p})$ given by (8) to be realizable, (5) tells us that K must exceed 0·4. If $K = 3/7$, $b_6 = c_6$ in (10) and the group delay given by (9) is maximally flat, but (as shown in Reference 1, Fig. 1) begins to droop when $\rho\omega$ exceeds about 2. If K is raised to 0·44, however, there is a slight rise in the group-delay/frequency characteristic at low frequencies, and the group-delay remains very close to its zero-frequency value until $\rho\omega$ exceeds 3. This group-delay is plotted against frequency in Fig. 2 ($\lambda = 0$). We have therefore decided that the best value of K to make the group-delay τ as constant as possible, for as wide a range of ω (given ρ) as possible, is 0·44. The group-delay unit in Fig. 2 is discussed below.

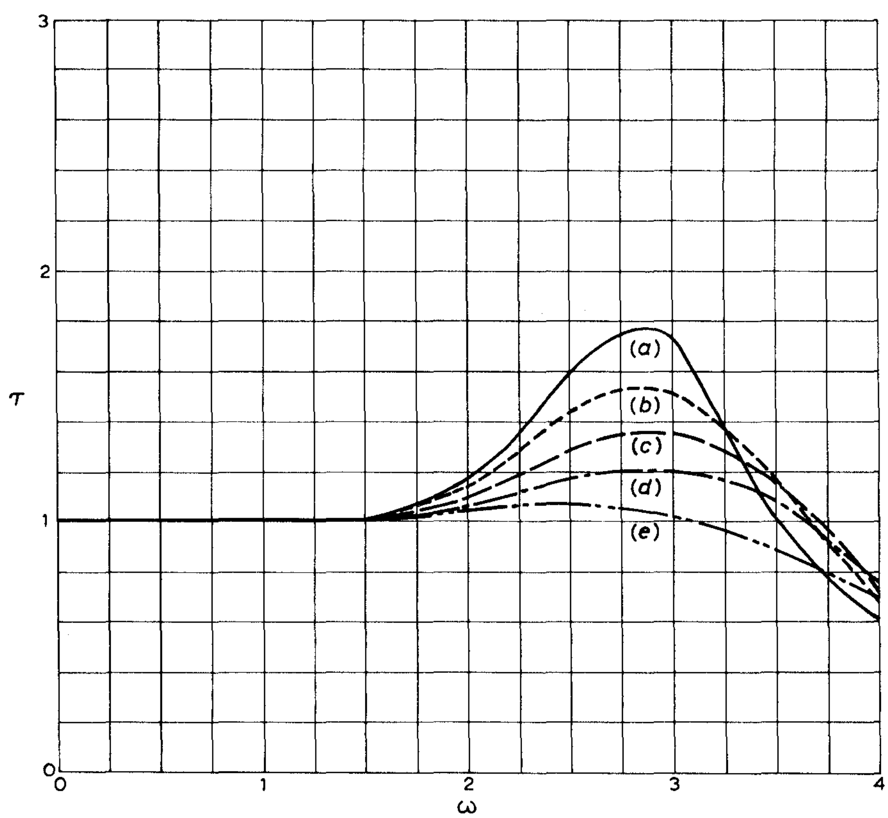

Fig. 2 - Variation of group delay with frequency when transfer function is given by Equations (11) and (12)

(a) $\lambda = 0{\cdot}012$ *(b)* $\lambda = 0{\cdot}009$ *(c)* $\lambda = 0{\cdot}006$ *(d)* $\lambda = 0{\cdot}003$ *(e)* $\lambda = 0$

3. COMPROMISE BETWEEN THE GOURIET AND BUTTERWORTH CASES

When $f(\mathrm{p})$ is given by (6), the variation of group delay with frequency is more than can often be tolerated. When $f(\mathrm{p})$ is given by (8) with $K = 0{\cdot}44$, however, the variation in amplitude is considerable but not intolerable. We now suppose that $f(\mathrm{p})$ is formed as the combination of, or compromise between, the Gouriet case with $\rho = 1$ and $K = 0{\cdot}44$ given by (8) and the Butterworth case given by (6) with $r = 1$, so that

$$1/f(\mathrm{p}) = 1 + d_1\mathrm{p} + d_2\mathrm{p}^2 + d_3\mathrm{p}^3 + d_4\mathrm{p}^4 \qquad (11)$$

where

$$\left.\begin{aligned} d_1 &= \{1 + 2\lambda(\sin\tfrac{\pi}{8} + \sin\tfrac{3\pi}{8})\}/(1+\lambda) \\ d_2 &= \{0{\cdot}44 + 2\lambda + 4\lambda\sin\tfrac{\pi}{8}\sin\tfrac{3\pi}{8}\}/(1+\lambda) \\ d_3 &= \{0{\cdot}44 - \tfrac{1}{3} + 2\lambda(\sin\tfrac{\pi}{8} + \sin\tfrac{3\pi}{8})\}/(1+\lambda) \\ d_4 &= \{\tfrac{0{\cdot}04}{3} + \lambda\}/(1+\lambda) \end{aligned}\right\} \qquad (12)$$

and λ is at our choice; making λ tend to zero gives the Gouriet case and making λ tend to infinity gives the Butterworth case. We now consider the general deterioration that sets in rapidly as λ is increased from zero, and seek to determine the best available compromise value. It is first necessary to check that there is no positive value of λ for which $f(\mathrm{p})$ given by (11) and (12) becomes unrealizable by means of passive networks. If a_i given by (1) is replaced by d_i given by (12), the inequality (5) is satisfied when $\lambda > -0{\cdot}13888$, and all the quantities d_i are positive if $\lambda > -0{\cdot}013333$, and therefore $f(\mathrm{p})$ given by (11) is realizable by means of a passive network for all positive λ which alone are relevant.

So far, we have paid some attention to variations in the amplitude and group-delay at frequencies where the amplitude is comparable with its value of unity at zero frequency but we have not considered the rapidity of cut-off at higher frequencies. In the Gouriet case with $\rho = 1$ and $K = 0{\cdot}44$, the variation of group-delay is very small until ω exceeds about 3, and the amplitude (which is unity at zero frequency) falls to about two-thirds when $\omega = 2{\cdot}5$ and to about a quarter when $\omega = 4$. If we regard the effective pass-band within which we are trying to make group-delay and amplitude as constant as possible as the range from $\omega = 0$ to $\omega = 2{\cdot}5$, the Gouriet case ($\lambda = 0$) is completely satisfactory from the point of view of group-delay (which is the reason for considering the case). The variations of amplitude within the effective pass-band are appreciable but not necessarily intolerable, and the rapidity of cut-off is reasonably satisfactory. It remains to see whether we can improve the amplitude variations by increasing λ without upsetting the satisfactory absence of appreciable group-delay variations.

The smaller value of λ which causes the amplitude to be unity when $\omega = 2{\cdot}5$ is $0{\cdot}0145$, but for this value of λ the group-delay variations when ω is between $2{\cdot}0$ and $2{\cdot}5$ are excessive. The corresponding value of λ such that the amplitude is $0{\cdot}9$ when $\omega = 2{\cdot}5$ is $0{\cdot}0109$, but even in this case the group-delay variations when ω is between 2 and $2{\cdot}5$ are excessive.

Fig. 1 gives the amplitude variations with frequency for $\lambda = 0$, $0{\cdot}003$, $0{\cdot}006$, $0{\cdot}009$ and $0{\cdot}012$, and Fig. 2 gives the corresponding group-delay variations. Equation 9 shows that the group delay in the Gouriet case [(e) or $\lambda = 0$ in Fig. 2] is ρ, which is unity since in Fig. 2 we have given both ρ and r the normalized value unity. The zero-frequency value of group delay for the other curves of Fig. 2 is d_1, given by the first of Equations 12 when $r = \rho = 1$, and this is only very slightly different from unity in all cases considered. Figures 1 and 2 suggest that the best compromise available for an insertion transfer function which is the reciprocal of a polynomial of degree 4 is when λ in Equation (12) is chosen in the range from 0 to $0{\cdot}012$. The exact value will depend upon the relative importance of group-delay constancy, amplitude constancy within the effective pass-band and rapidity of cut-off for the particular application involved.

4. STEP RESPONSES AND K-RATINGS

If unit-step input $H(t)$ (zero for negative t, unity for positive t) is applied to a low-pass filter having the transfer function given in Equation (1), the output or response $h(t)$ can be determined explicitly by a calculation which is tedious but not difficult. The results are given below:

(a) for the Butterworth case, where the transfer function $1/f(\mathrm{p})$ is given by Equation (6) with $r = 1$, the step response $h_1(t)$ is given by

$$h_1(t) = [1 + e^{-\lambda_1 t}\cos(\mu_1 t + \tfrac{\pi}{4}) - (1 + \sqrt{2})e^{-\mu_1 t}\cos(\lambda_1 t - \tfrac{\pi}{4})]\,H(t) \qquad (13)$$

where

$$\lambda_1 = \sin(\pi/8) = 0{\cdot}382683; \quad \mu_1 = \sin(3\pi/8) = 0{\cdot}923880 \qquad (14)$$

This is plotted in Fig. 3.

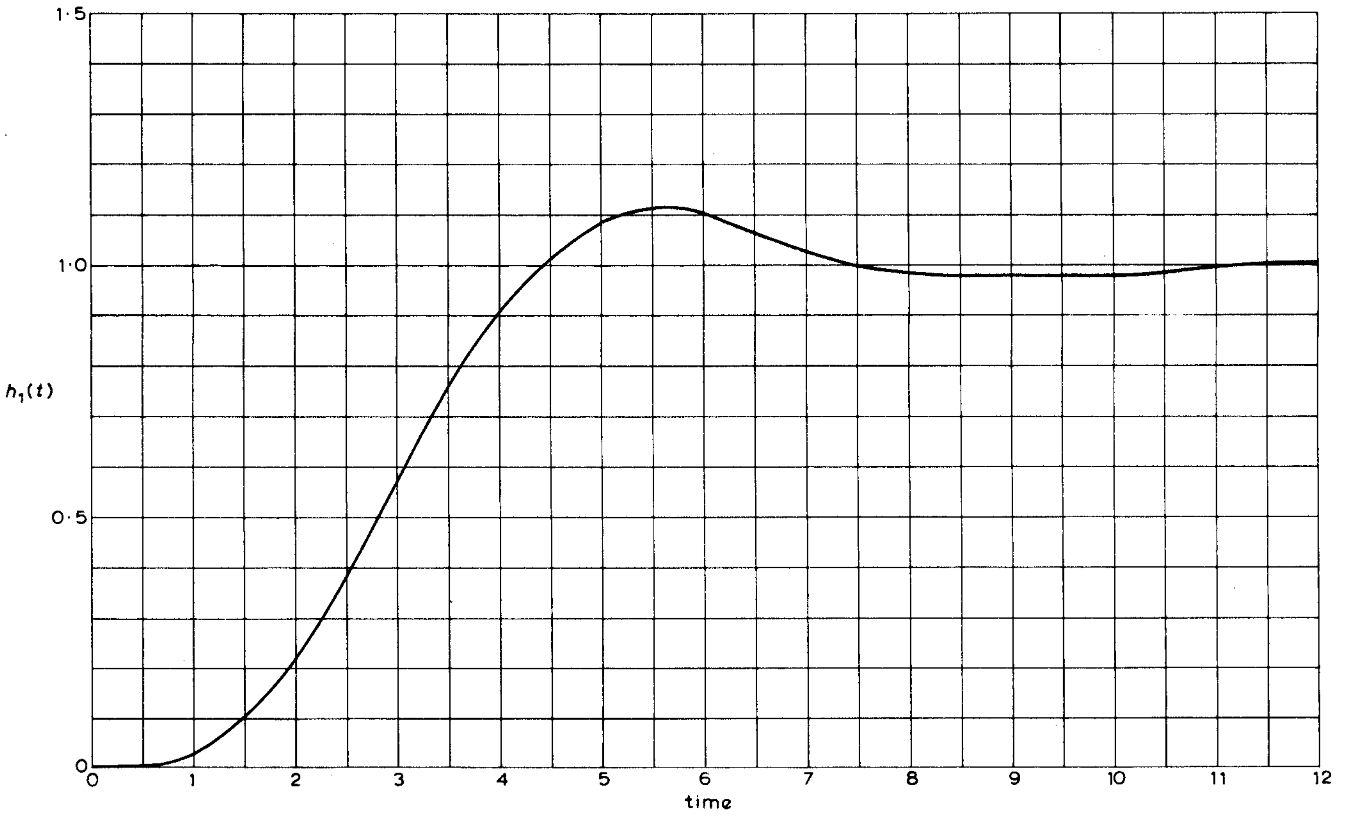

Fig. 3 - Step response in "Butterworth" case (Equation 13)

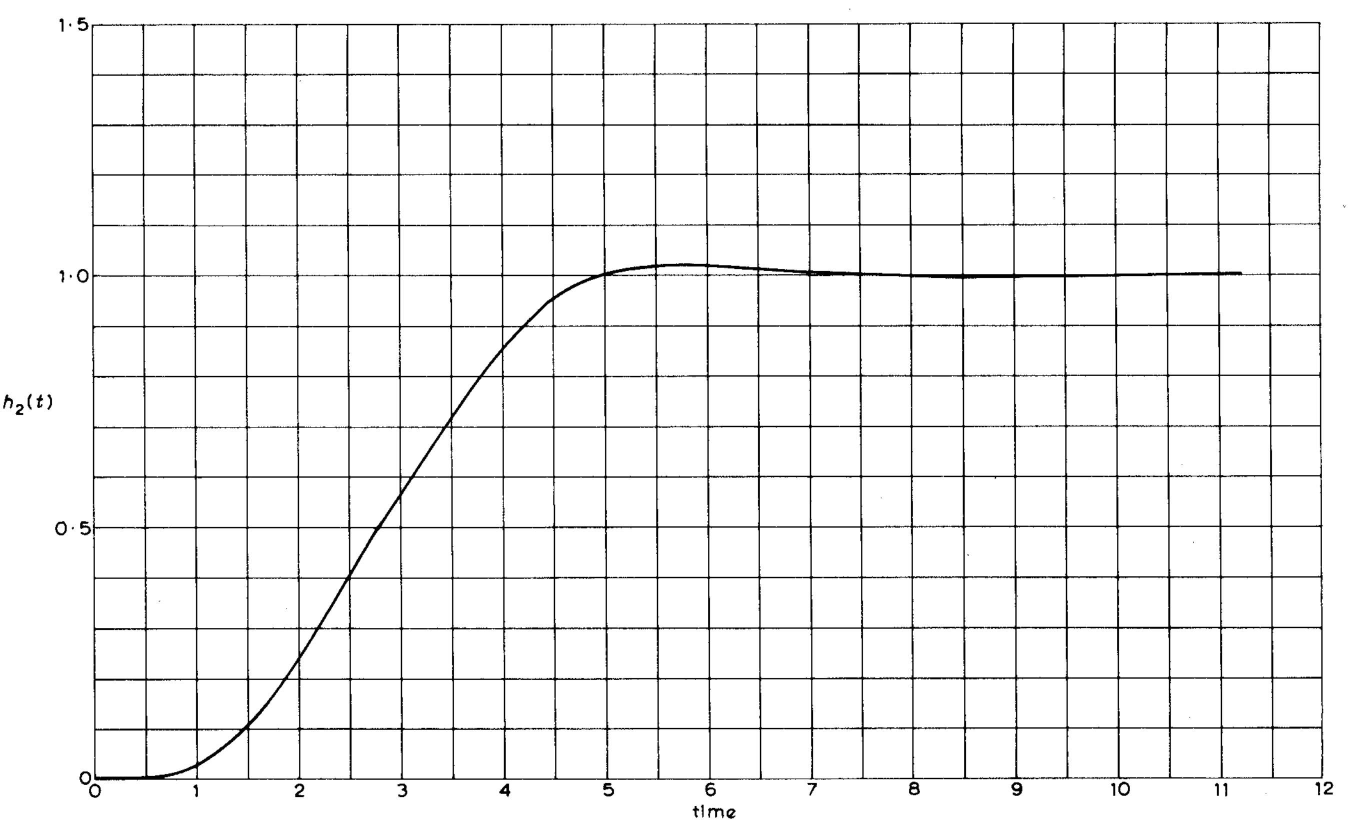

Fig. 4 - Step response in "Gouriet" case (Equation 15)

(b) for the Gouriet case, where the transfer function $1/f(\mathrm{p})$ is given by Equation (8) (with $\rho = 1$ and $K = 0{\cdot}44$), the step response $h_2(t)$ is given by

$$\begin{aligned} h_2(t) = \; & [\,1 + 0{\cdot}14407e^{-\lambda_2 t}\sin\mu_2 t + 0{\cdot}87295e^{-\lambda_2 t}\cos\mu_2 t \\ & - 3{\cdot}10876e^{-\nu_2 t}\sin\xi_2 t - 1{\cdot}87295e^{-\nu_2 t}\cos\xi_2 t\,]\, H(t) \end{aligned} \qquad (15)$$

where

$$\left.\begin{aligned} \lambda_2 &= 1{\cdot}54941 \qquad & \mu_2 &= 2{\cdot}92984 \\ \nu_2 &= 2{\cdot}45112 \qquad & \xi_2 &= 0{\cdot}90589 \end{aligned}\right\} \qquad (16)$$

This is plotted in Fig. 4.

(c) for the "compromise" case, where the transfer function $1/f(\mathrm{p})$ is given by Equations (11) and (12) (with $\lambda = 0{\cdot}009$ in Equation (12)) the step response $h_3(t)$ is

$$\begin{aligned} h(t) = \; & [\,1 - 0{\cdot}29951e^{-\lambda_3 t}\sin\mu_3 t + 0{\cdot}47492e^{-\lambda_3 t}\cos\mu_3 t \\ & - 2{\cdot}05647e^{-\nu_3 t}\sin\xi_3 t - 1{\cdot}47492e^{-\nu_3 t}\cos\xi_3 t\,]\, H(t) \end{aligned} \qquad (17)$$

where

$$\left.\begin{aligned} \lambda_3 &= 0{\cdot}85346 \qquad & \mu_3 &= 2{\cdot}88434 \\ \nu_3 &= 2{\cdot}06199 \qquad & \xi_3 &= 0{\cdot}86170 \end{aligned}\right\} \qquad (18)$$

This is plotted in Fig. 5.

All these expressions check satisfactorily in so far as

(a) $h_1(t)$, $h_2(t)$ and $h_3(t)$ all tend to unity when t tends to infinity. This is correct since each $f(\mathrm{p})$ from which $h_1(t)$ etc. were derived tends to unity as p tends to 0.

(b) $h_1(t)$, $h_2(t)$, $h_3(t)$ and their first three derivatives are all zero when $t = 0$. This is correct, since each $f(\mathrm{p})$ from which $h_1(t)$ etc. were derived is such that $\mathrm{p}^n f(\mathrm{p})$ tends to zero when p tends to infinity for $n = 1, 2, 3$.

(c) The fourth derivatives of $h_1(t)$, $h_2(t)$ and $h_3(t)$ are all unity when $t = 0$. This is correct since each $f(\mathrm{p})$ from which $h_1(t)$ etc. were derived is such that

$$\lim_{\mathrm{p}\to\infty} \mathrm{p}^4 f(\mathrm{p}) = 1$$

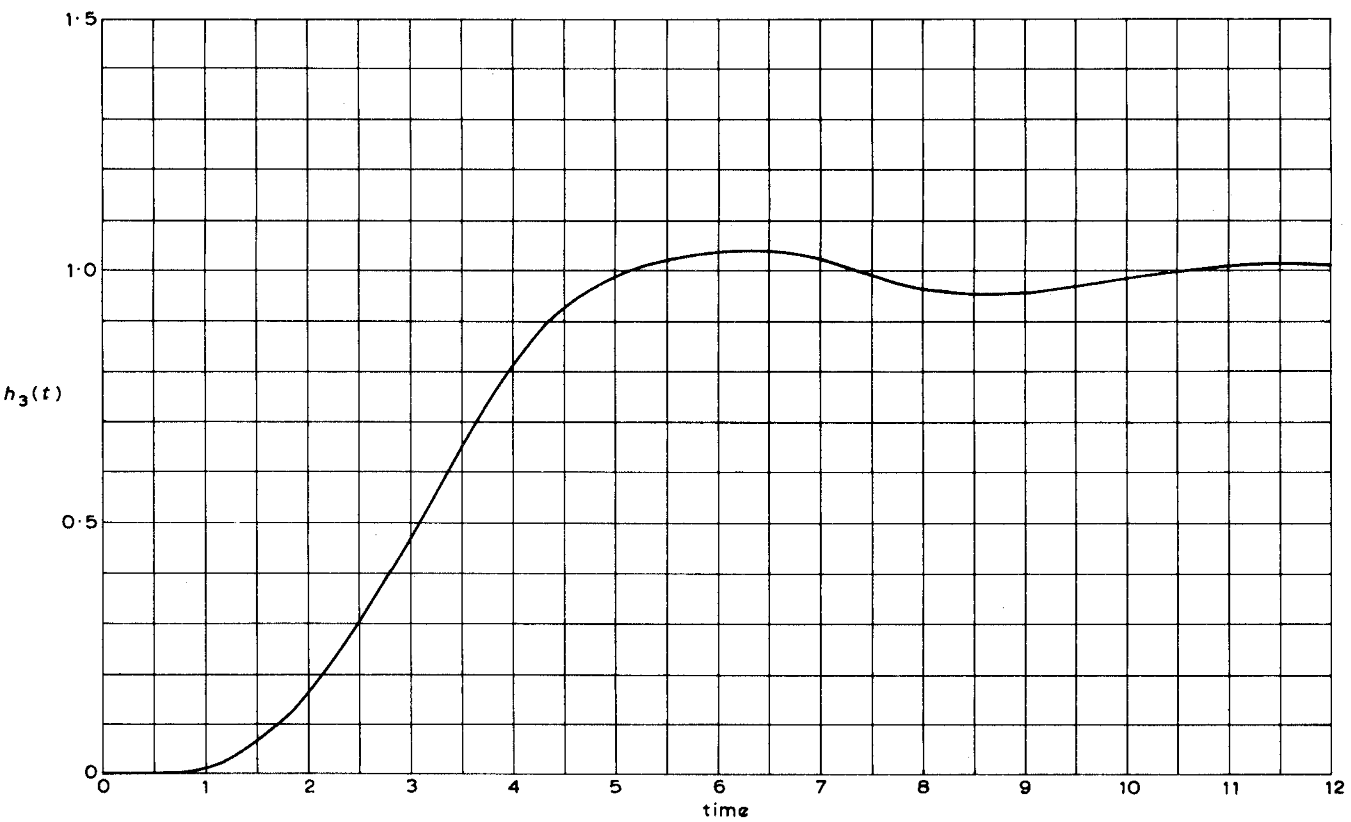

Fig. 5 - Step response in "Compromise" case (Equation 17)

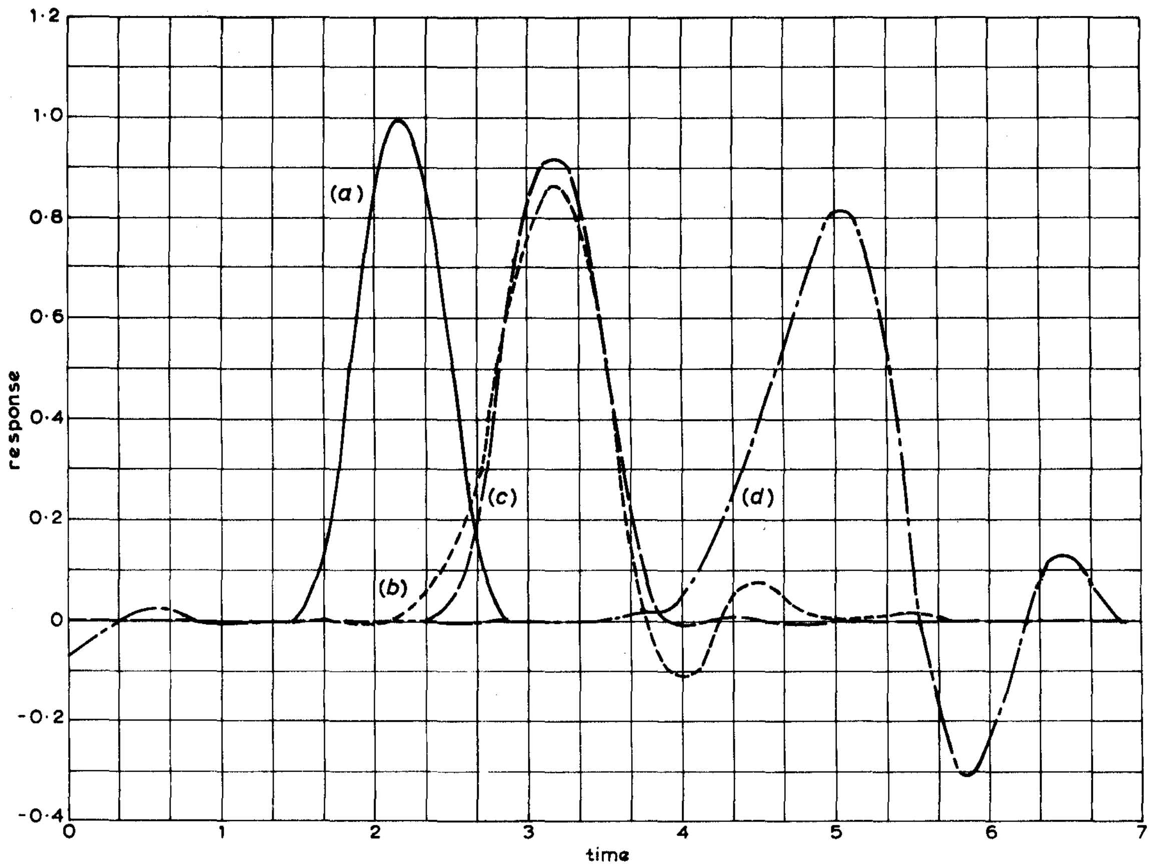

Fig. 6 - Responses to a 2T pulse

(*a*) 2T pulse (*b*) "Compromise" case (*c*) "Gouriet" case (*d*) "Butterworth" case

The response to a 2T pulse (associated with the frequency at which each of the filters above discussed is 3 dB down) was also obtained and is shown in Fig. 6. These responses were calculated by means of a computer programme* which also gives the following K-ratings:

For the Butterworth case: 10·66%

For the Gouriet case: 1·44%

For the Compromise case: 4·52%

5. CONCLUSIONS AND RECOMMENDATIONS

For a transfer function which is restricted to be the reciprocal of a polynomial of degree 4, it is not easy to find a good compromise to meet the conflicting requirements of satisfactory constancy of amplitude and group delay within the effective pass-band, and satisfactory cut-off properties. The best compromise available appears to be very close to the transfer function given by Equation (8) with K = 0·44 (called the Gouriet case in the foregoing).

* due to Messrs R.W. Lee and C.R.G. Reed

If the investigation is extended to transfer functions of higher degree, or the Butterworth case used here is replaced by a Tchebycheff-type case, a very similar result is to be expected.

If the main requirement is for simplicity rather than close tolerances, constancy of group-delay in the effective pass-band should be considered first; if the associated variation of amplitude (when the group-delay requirements are met) is unsatisfactory there is probably no alternative to the more usual procedure of meeting the amplitude requirements first by means of a minimum phase network, and subsequently correcting the group-delay by means of associated all-pass networks.

6. REFERENCE

1. GOURIET, G.G., 1957, Two theorems concerning group delay with practical application to delay correction. *Proc.Instn elect.Engrs,* 1958, **105C,** 7, pp. 240 - 244.

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Printed by BBC Research Department, Kingswood Warren, Tadworth, Surrey